IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Quasideterminant solutions of a non-Abelian Hirota–Miwa equation

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2007 J. Phys. A: Math. Theor. 40 12607

(http://iopscience.iop.org/1751-8121/40/42/S07)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.146
The article was downloaded on 03/06/2010 at 06:21

Please note that terms and conditions apply.

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 12607–12617 doi:10.1088/1751-8113/40/42/S07

Quasideterminant solutions of a non-Abelian Hirota–Miwa equation

C R Gilson[1], J J C Nimmo[1] and Y Ohta[2]

[1] Department of Mathematics, University of Glasgow, UK
[2] Department of Mathematics, Kobe University, Japan

Received 9 February 2007, in final form 19 April 2007
Published 2 October 2007
Online at stacks.iop.org/JPhysA/40/12607

Abstract
A non-Abelian version of the Hirota–Miwa equation is considered. In an earlier paper of Nimmo (2006 *J. Phys. A: Math. Gen.* **39** 5053–65) it was shown how solutions expressed as quasideterminants could be constructed for this system by means of Darboux transformations. In this paper, we discuss these solutions from a different perspective and show that the solutions are quasi-Plücker coordinates and that the non-Abelian Hirota–Miwa equation may be written as a quasi-Plücker relation. The special case of the matrix Hirota–Miwa equation is also considered using a more traditional, bilinear approach and the techniques are compared.

PACS number: 02.03.Ik
Mathematics Subject Classification: 35Q58, 39A10

1. Introduction

There has been interest recently in a noncommutative version of some of the well-known soliton equations including the KP and KdV equations [1–10]. In most of this recent work, the noncommutativity arises because of a quantization of the phase space in which independent variables do not commute and the (commutative) product of real- or complex-valued functions of these is replaced by the associative but noncommutative Moyal star product.

In many cases, the noncommutative version of the equation is obtained by considering the compatibility of the same Lax pair that is used in the commutative case. The only difference between the commutative and noncommutative cases being that the assumption that coefficients in the Lax pair commute is relaxed.

The Hirota–Miwa equation, or the discrete KP equation, [11, 12] is the fully discrete three-dimensional integrable system

$$(a_2 - a_3)\tau_{,1}\tau_{,23} + (a_3 - a_1)\tau_{,2}\tau_{,31} + (a_1 - a_2)\tau_{,3}\tau_{,12} = 0, \tag{1}$$

where a_k are constants, $\tau = \tau(n_1, n_2, n_3)$ and subscripts $_{,j}$ denote a forward shift in the corresponding discrete variables n_1, n_2 or n_3. This is the compatibility conditions for the linear system

$$\phi_{,i} - \phi_{,j} = U_{ij}\phi, \tag{2}$$

for $i < j = 1, 2, 3$, where

$$U_{ij} = (a_i - a_j)\frac{\tau_{,ij}\tau}{\tau_{,i}\tau_{,j}}. \tag{3}$$

If one considers the linear system (2) in a noncommutative setting, the parametrization (3) is not appropriate but nonetheless we obtain a nonlinear system for the U_{ij} which we call the non-Abelian Hirota–Miwa equation. This system was derived in [13] and solutions in terms of quasideterminants [14] were derived by means of Darboux transformations.

It has long been known [15] that equations in the KP hierarchy can be interpreted as Plücker relations on an infinite-dimensional Grassmann manifold. In this paper, we establish a result which hints at a corresponding situation in the noncommutative case as we show that the quasideterminant solutions are, in the terminology of Gelfand *et al* [14, 16], quasi-Plücker coordinates and the non-Abelian Hirota–Miwa equation is a quasi-Plücker relation.

This paper is arranged as follows. In section 2, the properties of quasideterminants that are used in the rest of the paper are described. In section 3, the main results from [13] are summarized and the quasideterminant solutions found there are presented. The next two sections give two contrasting approaches to direct verification of these solutions. In the first, applicable in the general non-Abelian case, the solutions are recognized to be quasi-Plücker coordinates and the direct verification is effected using quasi-Plücker relations and other properties. Finally, we consider an alternative approach which works only for the matrix Hirota–Miwa equation. In this it is shown how standard Plücker relations are used to verify the solutions.

2. Preliminaries

In this section, we will state the properties of quasideterminants that are used in this article. The reader is referred to the original papers [14, 16] for a more detailed description and proofs of the results described here.

2.1. Quasideterminants

For an $n \times n$ matrix $M = (a_{i,j})$ over a ring $\mathcal{R}$ (noncommutative, in general) there are n^2 *quasideterminants* written as $|M|_{i,j}$ for $i, j = 1, \ldots, n$ which are also elements of $\mathcal{R}$. They are defined recursively by

$$|M|_{i,j} = a_{i,j} - r_i^j (M^{i,j})^{-1} c_j^i, \qquad M^{-1} = \left(|M|_{j,i}^{-1}\right)_{i,j=1,\ldots,n}. \tag{4}$$

In the above, r_i^j represents the ith row of M with the jth element removed, c_j^i represents the jth column with the ith element removed and $M^{i,j}$ represents the submatrix obtained by removing the ith row and the jth column from A. Quasideterminants can also be denoted as shown below, by boxing the entry about which the expansion is made

$$|M|_{i,j} = \begin{vmatrix} M^{i,j} & c_j^i \\ r_i^j & \boxed{a_{i,j}} \end{vmatrix}.$$

In the case $n = 1$, let $M = (a)$, say, and then there is just one quasideterminant $|M|_{1,1} = |\boxed{a}| = a$. For $n = 2$, if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then there are four quasideterminants

$$|M|_{1,1} = \begin{vmatrix} \boxed{a} & b \\ c & d \end{vmatrix} = a - bd^{-1}c, \qquad |M|_{1,2} = \begin{vmatrix} a & \boxed{b} \\ c & d \end{vmatrix} = b - ac^{-1}d,$$

$$|M|_{2,1} = \begin{vmatrix} a & b \\ \boxed{c} & d \end{vmatrix} = c - db^{-1}a, \qquad |M|_{2,2} = \begin{vmatrix} a & b \\ c & \boxed{d} \end{vmatrix} = d - ca^{-1}b.$$

From this, we can obtain the matrix inverse

$$M^{-1} = \begin{pmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{pmatrix},$$

which is then used in the definition of the nine quasideterminants of a 3×3 matrix.

To explain results in the simplest way, we will sometimes write M as a block matrix

$$M = \begin{pmatrix} A & B \\ C & d \end{pmatrix},$$

where $d \in \mathcal{R}$, A is a square matrix over $\mathcal{R}$ of arbitrary size and B, C are column and row vectors, respectively, over $\mathcal{R}$ of compatible lengths. For this matrix,

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix} = d - CA^{-1}B. \tag{5}$$

2.2. Commutative cases

If the entries in M commute then the above becomes the familiar formula for the inverse of a 2×2 matrix with entries expressed as ratios of determinants. Indeed this is true in general; if the entries in M commute then

$$|M|_{i,j} = \frac{\begin{vmatrix} M^{i,j} & c^i_j \\ r^j_i & a_{i,j} \end{vmatrix}}{|M^{i,j}|} = (-1)^{i+j} \frac{|M|}{|M^{i,j}|}, \tag{6}$$

the alternating sign arising from the reordering of rows and columns in the numerator.

In the special case that the ring $\mathcal{R}$ is the (noncommutative) ring of $N \times N$ matrices over another commutative ring, the (i, j) quasideterminant of M, $|M|_{i,j}$ is itself an $N \times N$ matrix with (k, l)th entry

$$(|M|_{i,j})_{k,l} = \begin{vmatrix} M^{i,j} & \left(c^i_j\right)^k_l \\ \left(r^j_i\right)^l_k & \boxed{(a_{i,j})_{k,l}} \end{vmatrix}. \tag{7}$$

2.3. Invariance under row and column operations

The quasideterminants of a matrix have invariance properties similar to those of determinants under elementary row and column operations applied to the matrix. Consider the following quasideterminant of an $n \times n$ matrix;

$$\left| \begin{pmatrix} E & 0 \\ F & g \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \right|_{n,n} = \begin{vmatrix} EA & EB \\ FA + gC & FB + gd \end{vmatrix}_{n,n} = g(d - CA^{-1}B) = g \begin{vmatrix} A & B \\ C & d \end{vmatrix}_{n,n}. \tag{8}$$

This formula describes the effect on the quasideterminants of a matrix of performing elementary row operations, involving left multiplication, on the matrix. The zero block in the first matrix means that operations which add left-multiples of the row containing the expansion point to other rows are excluded from consideration, but all other elementary row operations are included. We see that left-multiplying the row containing the expansion point by g has the effect of left-multiplying the quasideterminant by g and that all other operations leave the quasideterminant unchanged. There is analogous invariance under column operations involving multiplication on the right.

2.4. Noncommutative Jacobi identity

There is a quasideterminant version of Jacobi's identity for determinants, called the noncommutative Sylvester's theorem by Gelfand and Retakh [16]. The simplest version of this identity is given by

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}. \tag{9}$$

2.5. Quasi-Plücker coordinates

Given an $(n+k) \times n$ matrix A, denote the ith row of A by A_i, the submatrix of A having rows with indices in a subset I of $\{1, 2, \ldots, n+k\}$ by A_I and $A_{\{1,\ldots,n+k\}\setminus\{i\}}$ by $A_{\hat{i}}$. Given $i, j \in \{1, 2, \ldots, n+k\}$ and I such that $\#I = n-1$ and $j \notin I$, one defines the *(right) quasi-Plücker coordinates*

$$r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix} A_I \\ A_i \end{vmatrix}_{ns} \begin{vmatrix} A_I \\ A_j \end{vmatrix}_{ns}^{-1} = - \begin{vmatrix} A_I & 0 \\ A_i & \boxed{0} \\ A_j & 1 \end{vmatrix}, \tag{10}$$

for any column index $s \in \{1, \ldots, n\}$. The final equality in (10) comes from an identity of the form (9) and proves that the definition is independent of the choice of s.

The following is an easy consequence of the definition:

$$r_{ij}^I = 0 \quad \text{if} \quad i \in I, \text{(and is not defined if } j \text{ were in } I) \tag{11}$$

$$r_{ii}^I = 1 \tag{12}$$

$$r_{ji}^I = \left(r_{ij}^I\right)^{-1} \tag{13}$$

$$r_{ij}^I r_{jk}^I = r_{ik}^I. \tag{14}$$

Remark. A useful consequence of (10) and (13) is the identity

$$\begin{vmatrix} A^I & 0 \\ A^i & \boxed{0} \\ A^j & 1 \end{vmatrix}^{-1} = \begin{vmatrix} A^I & 0 \\ A^i & 1 \\ A^j & \boxed{0} \end{vmatrix}, \tag{15}$$

which shows that quasideterminants of this form may be inverted very simply.

There are also the following properties which are proved in [16], *skew-symmetry*,

$$r_{ij}^{I\backslash\{i,j\}} r_{jk}^{I\backslash\{j,k\}} = -r_{ik}^{I\backslash\{i,k\}}, \tag{16}$$

for any $\{i, j, k\} \in I$ and *quasi-Plücker relations*

$$\sum_{j \in L} r_{ij}^{L\backslash\{j\}} r_{ji}^{M} = 1, \tag{17}$$

for any $i \notin M$, where $L, M \subset \{1, \ldots, n+k\}$ such that $\#L - 1 = \#M = n - 1$.

3. A non-Abelian Hirota–Miwa equation

This section is mainly a review of the results found in [13]. When discussing lattice equations one generally thinks of the dependent variable as a real- or complex-valued function of one or more integer-valued variables. Here we wish to abstract this notion as far as possible and think of a lattice in a more general way which encompasses the usual types of lattice systems. Similar ideas were also explored by Matveev in [17]. We suppose that dependent variables take values in a ring $\mathcal{R}$ and the lattice is defined by means of mappings $s_i : \mathcal{R} \to \mathcal{R}, i \in \mathbb{N}$, which we think of as moving between neighbouring sites on the (possibly infinite-dimensional) lattice. The particular examples we have in mind for these mappings include the usual shift operators $s_i(\phi(n_i)) = \phi(n_i + 1)$ or q-shift operators $s_i(\phi(n_i)) = \phi(q_i n_i)$, but also Darboux or Bäcklund transformations $s_i(\phi) = \tilde{\phi}$ mapping between different solutions of the lattice equations.

With these sorts of examples in mind, we require that the s_i have the following properties:

(1) all s_i are invertible and mutually commute, so that motion on the lattice is reversible and path independent;
(2) each s_i is a linear homomorphism so that the same (linear and nonlinear) relations satisfied at one lattice site are satisfied at all lattice sites,

and we refer to such mappings as *lattice operators*. We will also use the shorthand notation $X_{,i} = s_i(X)$ and $X_{,\bar{i}} = s_i^{-1}(X)$.

Now consider lattice operators s_i, $(i \in \mathbb{N})$, and invertible $U_{ij} \in \mathcal{R}$ and the system of linear equations for ϕ

$$\phi_{,i} - \phi_{,j} + U_{ij}\phi := s_i(\phi) - s_j(\phi) + U_{ij}\phi = 0, \qquad (i, j \in \mathbb{N}). \tag{18}$$

This system is algebraically over-determined and so for any i, j, k,

$$U_{ij} + U_{jk} + U_{ki} = 0. \tag{19}$$

This includes degenerate cases which arise when some of i, j, k coincide, namely $U_{ii} = 0$ and $U_{ij} + U_{ji} = 0$.

The compatibility conditions $\phi_{,ij} = \phi_{,ji}$ give the nonlinear conditions

$$U_{ij,k} U_{ik} = U_{ik,j} U_{ij}, \tag{20}$$

for i, j, k distinct (if any of i, j, k coincides this is an empty condition) together with

$$U_{ij,k} + U_{jk,i} + U_{ki,j} = 0. \tag{21}$$

By introducing $V_{ij} = U_{ij,\overline{ij}}$, one may rewrite (21) as the 'dual' equation

$$V_{ij} + V_{jk} + V_{ki} = 0, \tag{22}$$

and one may also rewrite (20) in terms of V_{ij} as

$$V_{ij} V_{ik,\bar{j}} = V_{ik} V_{ij,\bar{k}}. \tag{23}$$

It is straightforward to show that the U_{ij} satisfy (19) and (20) if and only if the V_{ij} satisfy (22) and (23) and so it suffices to consider either the equations for U_{ij}

$$U_{ij} + U_{jk} + U_{ki} = 0, \qquad U_{ij,k}U_{ik} = U_{ik,j}U_{ij}, \tag{24}$$

or for V_{ij}

$$V_{ij} + V_{jk} + V_{ki} = 0, \qquad V_{ik}V_{ij,\bar{k}} = V_{ij}V_{ik,\bar{j}}. \tag{25}$$

We call either of these equivalent systems the *non-Abelian Hirota–Miwa equation*.

If we fix on one particular three-dimensional sublattice (defined by s_1, s_2, s_3 say), the linear equations

$$\phi_{,i} - \phi_{,j} + U_{ij}\phi = 0, \qquad (i, j \in \{1, 2, 3\}) \tag{26}$$

are compatible provided (24) or (25) are satisfied for $\{i, j, k\} \subset \{1, 2, 3\}$. The remaining dimensions in the lattice defined by $s_4, s_5, \ldots = d_1, d_2, \ldots$ can be interpreted as Darboux transformations. In [13], these Darboux transformations were exploited in order to construct solutions of the non-Abelian Hirota–Miwa system expressed in terms of quasideterminants. Finally in this section, we will state these results.

Making the ansatz

$$U_{ij} = X_{i,j}a_{ij}X_i^{-1}, \tag{27}$$

where $X_i \in \mathcal{R}$ are invertible and $a_{ij} \in \mathcal{R}$, the nonlinear conditions (20) are satisfied for any X_i provided the a_{ij} themselves satisfy this condition. We also take $a_{ii} = 0$ so that $U_{ii} = 0$. With these choices, (23) are solved by writing

$$V_{ij} = Y_i^{-1}a_{ij}Y_{i,\bar{j}}, \tag{28}$$

where $Y_i = X_{i,\bar{i}}^{-1}$. In the commutative case, the skew-symmetry conditions

$$0 = U_{ij} + U_{ji} = X_{i,j}a_{ij}X_i^{-1} + X_{j,i}a_{ji}X_j^{-1} = a_{ij}\frac{X_{i,j}}{X_i} + a_{ji}\frac{X_{j,i}}{X_j},$$

can be solved identically by the additional ansatz $X_i = \tau_{,i}/\tau$ and taking $a_{ij} + a_{ji} = 0$. The remaining equation in (24) now becomes the standard Hirota–Miwa equation (1) and it is for this reason that we refer to (24) and (25) as non-Abelian Hirota–Miwa equations.

Now let X_i, $i = 1, 2, 3$, and $a_{ij}i, j \in \{1, 2, 3\}$ construct, through (27), a vacuum solution U_{ij} of (26). For example, one could choose $X_1 = X_2 = X_3 = 1$ and $a_{ij} = \alpha_i - \alpha_j$. Now choose n vacuum eigenfunctions, $\theta_1, \ldots, \theta_n$, i.e. solutions of (26) with the vacuum choice for U_{ij} made above. From these, one may construct the row vector $\theta = (\theta_1, \ldots, \theta_n)$ satisfying

$$s_i(\theta) - s_j(\theta) + U_{ij}\theta = 0. \tag{29}$$

Then, after n applications of Darboux transformations, the solution obtained is $\tilde{U}_{ij} = \tilde{X}_{i,j}a_{ij}\tilde{X}_i^{-1}$, where [13]

$$\tilde{X}_i = \begin{vmatrix} \theta & 1 \\ s_i(\theta) & 0 \\ \vdots & \vdots \\ s_i^{n-1}(\theta) & 0 \\ s_i^n(\theta) & \boxed{0} \end{vmatrix} X_i. \tag{30}$$

4. Direct verification of solutions using quasideterminants

In this section, we use the properties of quasideterminants stated in section 2 to effect a direct verification of the solutions described in (30). By recognizing that the quasideterminant in this equation is a quasi-Plücker coordinate (10), it may immediately be inverted using (15) to obtain

$$\tilde{X}_i^{-1} = X_i^{-1} \begin{vmatrix} \theta & \boxed{0} \\ s_i(\theta) & 0 \\ \vdots & \vdots \\ s_i^{n-1}(\theta) & 0 \\ s_i^n(\theta) & 1 \end{vmatrix}. \tag{31}$$

Then, using the linear equation (29) and the invariance properties (8), we get

$$\tilde{U}_{ij} = \begin{vmatrix} s_j(\theta) & 1 \\ s_j(s_i(\theta)) & 0 \\ \vdots & \vdots \\ s_j(s_i^{n-1}(\theta)) & 0 \\ s_j(s_i^n(\theta)) & \boxed{0} \end{vmatrix} \begin{vmatrix} s_j(\theta) & \boxed{0} \\ s_i(\theta) & 0 \\ \vdots & \vdots \\ s_i^{n-1}(\theta) & 0 \\ s_i^n(\theta) & 1 \end{vmatrix}, \qquad \tilde{V}_{ij} = \begin{vmatrix} s_i^{-1}(\theta) & 1 \\ \theta & 0 \\ \vdots & \vdots \\ s_i^{n-2}(\theta) & 0 \\ s_i^{n-1}(\theta) & \boxed{0} \end{vmatrix} \begin{vmatrix} s_i^{-1}(\theta) & \boxed{0} \\ s_j^{-1}(\theta) & 0 \\ \theta & 0 \\ \vdots & \vdots \\ s_i^{n-3}(\theta) & 0 \\ s_i^{n-2}(\theta) & 1 \end{vmatrix}. \tag{32}$$

In fact, this is the first key step in the direct verification, we can go one step further with $\tilde{V}_{ij}$. By making use of (29) and (8) we may write all but three rows of the quasideterminants in a way which is independent of the particular choices of i and j. Let $X^{(m)}$ denote $s_k^m(X)$ for any choice of $k = 1, 2$ or 3. Then it is straightforward to verify using (29) and (8) that

$$\tilde{V}_{ij} = \begin{vmatrix} \theta_{,\bar{\imath}} & 1 \\ \theta & 0 \\ \vdots & \vdots \\ \theta^{(n-3)} & 0 \\ \theta^{(n-2)} & 0 \\ \theta^{(n-1)} & \boxed{0} \end{vmatrix} \begin{vmatrix} \theta_{,\bar{\imath}} & \boxed{0} \\ \theta_{,\bar{\jmath}} & 0 \\ \theta & 0 \\ \vdots & \vdots \\ \theta^{(n-3)} & 0 \\ \theta^{(n-2)} & 1 \end{vmatrix}. \tag{33}$$

To complete the direct verification of these solutions we must show that they satisfy

$$V_{ii} = 0, \qquad V_{ij} + V_{ji} = 0, \qquad V_{ij} + V_{jk} + V_{ki} = 0. \tag{34}$$

One consequence of (8) is that a quasideterminant in which the row containing the expansion point is the same as another row is zero. The second factor in $\tilde{V}_{ii}$ is such a quasideterminant and so $\tilde{V}_{ii} = 0$. Next define the $(n+3) \times n$ matrix

$$\Theta = \begin{bmatrix} \theta_{,\bar{1}} \\ \theta_{,\bar{2}} \\ \theta_{,\bar{3}} \\ \theta^{(n-2)} \\ \theta^{(n-1)} \\ \theta \\ \theta^{(1)} \\ \vdots \\ \theta^{(n-3)} \end{bmatrix},$$

which will play the role of the matrix A in the definition of quasi-Plücker coordinates (10), and let $I = \{6, \ldots, n+3\}$ so that

$$\Theta_I = \begin{bmatrix} \theta \\ \theta^{(1)} \\ \vdots \\ \theta^{(n-3)} \end{bmatrix}.$$

Then the quasideterminants appearing in (33) can be expressed as quasi-Plücker coordinates,

$$r_{5,i}^{I\cup\{4\}}(\Theta) = -\begin{vmatrix} \theta & 0 \\ \vdots & \vdots \\ \theta^{(n-3)} & 0 \\ \theta^{(n-2)} & 0 \\ \theta^{(n-1)} & \boxed{0} \\ \theta_{\bar{i}} & 1 \end{vmatrix}, \qquad r_{i,4}^{I\cup\{j\}}(\Theta) = -\begin{vmatrix} \theta & 0 \\ \vdots & \vdots \\ \theta^{(n-3)} & 0 \\ \theta_{\bar{j}} & 0 \\ \theta_{\bar{i}} & \boxed{0} \\ \theta^{(n-2)} & 1 \end{vmatrix}.$$

Hence,

$$\tilde{V}_{ij} = r_{5,i}^{I\cup\{4\}} r_{i,4}^{I\cup\{j\}}, \tag{35}$$

where here and from now on we omit reference to the matrix Θ. Properties (14), (16) and (17) now take the particular form

$$r_{a,b}^{I\cup\{d\}} r_{b,c}^{I\cup\{d\}} = r_{a,c}^{I\cup\{d\}} \tag{36}$$

$$r_{a,b}^{I\cup\{c\}} r_{b,c}^{I\cup\{a\}} = -r_{a,c}^{I\cup\{b\}} \tag{37}$$

$$r_{a,b}^{I\cup\{c\}} r_{b,a}^{I\cup\{d\}} + r_{a,c}^{I\cup\{b\}} r_{c,a}^{I\cup\{d\}} = 1. \tag{38}$$

Using (36) and (37),

$$\begin{aligned} \tilde{V}_{ij} + \tilde{V}_{ji} &= r_{5,i}^{I\cup\{4\}} r_{i,4}^{I\cup\{j\}} + r_{5,j}^{I\cup\{4\}} r_{j,4}^{I\cup\{i\}} \\ &= r_{5,i}^{I\cup\{4\}} \left(r_{i,4}^{I\cup\{j\}} + r_{i,j}^{I\cup\{4\}} r_{j,4}^{I\cup\{i\}} \right) \\ &= r_{5,i}^{I\cup\{4\}} \left(r_{i,4}^{I\cup\{j\}} - r_{i,4}^{I\cup\{j\}} \right) \\ &= 0. \end{aligned}$$

Similarly, using the quasi-Plücker relation (38) as well as (36) and (37), we have

$$\begin{aligned}\tilde{V}_{12}+\tilde{V}_{23}+\tilde{V}_{31} &= r_{5,1}^{I\cup\{4\}}r_{1,4}^{I\cup\{2\}}+r_{5,2}^{I\cup\{4\}}r_{2,4}^{I\cup\{3\}}+r_{5,3}^{I\cup\{4\}}r_{3,4}^{I\cup\{1\}}\\ &= r_{5,1}^{I\cup\{4\}}\left(r_{1,4}^{I\cup\{2\}}+r_{1,3}^{I\cup\{4\}}r_{3,2}^{I\cup\{4\}}r_{2,4}^{I\cup\{3\}}+r_{1,3}^{I\cup\{4\}}r_{3,4}^{I\cup\{1\}}\right)\\ &= r_{5,1}^{I\cup\{4\}}\left(r_{1,4}^{I\cup\{2\}}-r_{1,3}^{I\cup\{4\}}r_{3,4}^{I\cup\{2\}}-r_{1,4}^{I\cup\{3\}}\right)\\ &= r_{5,1}^{I\cup\{4\}}\left(1-r_{1,3}^{I\cup\{4\}}r_{3,1}^{I\cup\{2\}}-r_{1,4}^{I\cup\{3\}}r_{4,1}^{I\cup\{2\}}\right)r_{1,4}^{I\cup\{2\}}\\ &= 0,\end{aligned}$$

and so the verification is complete.

This verification gives a suggestion that quasi-Plücker relations play a similarly important role in direct verification of solutions in the noncommutative situation as standard Plücker relations do in the commutative case and we intend to pursue this idea more fully in future work.

5. An alternative method for the matrix Hirota–Miwa equation

In the case that $\mathcal{R}$ is the ring of $r\times r$ complex matrices, an alternative, more familiar direct approach is available. This involves expressing the matrix entries as ratios of determinants, re-expressing system (25) in bilinear form and verifying the solutions using identities arising from Laplace expansions. In this section, we will contrast this more familiar, bilinear approach with the direct verification using quasideterminants described in the last section. We will see that the proof has a similar sequence of steps but, in contrast with the proof involving quasideterminants, has a rather ad hoc feel to it, in which the details of the calculation such as keeping track of the correct power of -1 are rather intricate.

In the case that we seek solutions V_{ij} of (25) has solutions in this matrix ring, the ingredients in solutions (33), the elements θ_k are themselves also (invertible) $r\times r$ matrices and the concatenation of these, θ, is an $r\times rn$ matrix. This expression for solutions (33) as quasideterminants can then be re-expressed in terms of a product of matrices whose entries are ratios of $n\times n$ determinants,

$$V_{ij}=\frac{G_i}{F_i}\frac{h_{ij}}{F_j},\tag{39}$$

where for each i, j, F_i is scalar and G_i and h_{ij} are $r\times r$ matrices. For a more compact notation, allowing determinants to be expressed in terms of columns rather than rows, θ is replaced by its transpose so that now θ is an $rn\times r$ matrix. The specific expressions are

$$F_i=\left|s_i^{-1}(\theta)\quad\theta\quad\theta^{(1)}\quad\cdots\quad\theta^{(n-2)}\right|,\tag{40}$$

and the (p,q)th entries of G_i and h_{ij} are

$$(G_i)_{pq}=(-1)^{rn+1+q}\left|s_i^{-1}(\theta)_{\hat{q}}\quad\theta\quad\theta^{(1)}\quad\cdots\quad\theta^{(n-2)}\quad\theta_p^{(n-1)}\right|,\tag{41}$$

and

$$(h_{ij})_{pq}=(-1)^{r+q}\left|s_i^{-1}(\theta)_p\quad s_j^{-1}(\theta)\quad\theta\quad\cdots\quad\theta^{(n-3)}\quad\theta_{\hat{q}}^{(n-2)}\right|,\tag{42}$$

respectively, where θ_p denotes the pth column of θ and $\theta_{\hat{q}}$ means all the columns of θ except the qth.

If V_{ij} is to satisfy (34) then we must show that

$$h_{ii}=0,\qquad G_ih_{ij}+G_jh_{ji}=0,\qquad G_ih_{ij}F_k+G_jh_{jk}F_i+G_kh_{ki}F_j=0.\tag{43}$$

The first of these is obvious since if $j=i$ then two columns in h_{ii} are the same. We will now show how the other two conditions may be verified.

For each p, q consider the $2rn \times 2rn$ determinants

$$L_{ij} = \left| \begin{array}{ccccc|ccccc|c} \theta & \theta^{(1)} & \cdots & \theta^{(n-2)} & \theta_p^{(n-1)} & 0 & 0 & \cdots & 0 & 0 & s_i^{-1}(\theta) \\ \hline 0 & 0 & \cdots & 0 & 0 & s_j^{-1}(\theta) & \theta & \cdots & \theta^{(n-3)} & \theta_{\hat{q}}^{(n-2)} & s_i^{-1}(\theta) \end{array} \right|.$$

The expansion of L_{ij} by $rn \times rn$ minors is, up to a sign independent of i and j,

$$\sum_{m=1}^{r} (G_i)_{pm}(h_{ij})_{mq} = (G_i h_{ij})_{pq}.$$

By using elementary row and column operations we also see that $L_{ji} = -L_{ij}$, and so we have verified the second conditions

$$G_i h_{ij} + G_j h_{ji} = 0. \tag{44}$$

For the final condition, it is necessary to introduce auxiliary matrix variables J_{ij}, where

$$(J_{ij})_{pq} = (-1)^q \left| s_i^{-1}(\theta)_{\hat{q}} \quad s_j^{-1}(\theta)_p \quad \theta \quad \cdots \quad \theta^{(n-2)} \right|. \tag{45}$$

By a similar expansion, the determinants

$$M_{ij} = \left| \begin{array}{cccc|c|cccc} s_i^{-1}(\theta)_{\hat{q}} & \theta & \cdots & \theta^{(n-2)} & s_j^{-1}(\theta) & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & s_j^{-1}(\theta) & \theta & \cdots & \theta^{(n-2)} & \theta_p^{(n-1)} \end{array} \right| = (-1)^q (G_j J_{ij})_{pq}.$$

This determinant can also be rewritten as

$$(-1)^{rn+1} \left| \begin{array}{cccc|c|cccc} s_i^{-1}(\theta)_{\hat{q}} & \theta & \cdots & \theta^{(n-2)} & \theta_p^{(n-1)} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & \theta_p^{(n-1)} & s_j^{-1}(\theta) & \theta & \cdots & \theta^{(n-2)} \end{array} \right| = (-1)^q (G_i F_j)_{pq}.$$

So we get

$$G_j J_{ij} = G_i F_j, \tag{46}$$

for any i, j.

Finally, consider expansion of the determinants

$$\left| \begin{array}{ccccc|ccc|cccc} s_i^{-1}\theta & \theta & \cdots & \theta^{(n-3)} & \theta_{\hat{q}}^{(n-2)} & s_k^{-1}\theta & \left(s_j^{-1}\theta\right)_p & (\theta^{(n-2)})_q & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 & s_k^{-1}\theta & \left(s_j^{-1}\theta\right)_p & (\theta^{(n-2)})_q & \theta & \cdots & \theta^{(n-3)} & \theta_{\hat{q}}^{(n-2)} \end{array} \right|$$
$$= (-J_{kj}h_{ki} + h_{ji}F_k - F_i h_{jk})_{pq}.$$

After elementary row and column operations, this determinant can also be shown to vanish and so we get

$$-J_{kj}h_{ki} + h_{ji}F_k - h_{jk}F_i = 0. \tag{47}$$

Using (44) and (46) and then (47) it follows that

$$G_i h_{ij} F_k + G_j h_{jk} F_i + G_k h_{ki} F_j = G_j(-h_{ji}F_k + h_{jk}F_i + J_{kj}h_{ki}) = 0.$$

This completes the alternative direct verification in the matrix case.

6. Conclusions

In this paper, we have considered two direct approaches to verifying quasideterminant solutions of a non-Abelian Hirota–Miwa equation found in [13]. In the first approach, the solutions are expressed in terms of quasi-Plücker coordinates, and the verification is achieved by using properties of these objects, including quasi-Plücker relations. The second approach, available only in the matrix case and not involving quasideterminants, uses an expression for the solutions as a matrix of ratios of determinants. In the first approach, the machinery used is more sophisticated but the verification is quite straightforward and applicable whatever the nature of the non-Abelian equation, matrix or quaternion or otherwise. The second verification needs less sophisticated tools (transformation to a bilinear form and standard Plücker relations) and is not always available since determinants are not defined in the general non-Abelian case. A direct quasideterminant approach has also been applied to noncommutative versions of the KP and mKP equations [10, 18], and it is hoped that this can be developed into a more widely applied direct approach to noncommutative integrable systems.

References

[1] Kupershmidt B A 2000 KP or mKP *Mathematical Surveys and Monographs* vol 78 (Providence, RI: American Mathematical Society)
[2] Paniak L D 2001 Exact noncommutative KP and KdV multi-solitons *Preprint* hep-th/0105185
[3] Sakakibara M 2004 *J. Phys.: Condens. Matter* A **37** L599
[4] Wang N and Wadati M 2003 *J. Phys. Soc. Japan* **72** 1366
[5] Wang N and Wadati M 2003 *J. Phys. Soc. Japan* **72** 1881
[6] Wang N and Wadati M 2003 *J. Phys. Soc. Japan* **72** 3055
[7] Hamanaka M 2003 Noncommutative solitons and *D*-branes *PhD thesis Preprint* hep-th/0303256
[8] Hamanaka M and Toda K 2003 *Phys. Lett.* A **316** 77
[9] Dimakis A and Müller-Hoissen F 2005 *J. Phys. A: Math. Theor.* **40** 3839
[10] Gilson C R and Nimmo J J C 2007 *J. Phys. A: Math. Theor.* **40** 3839
[11] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785
[12] Miwa T 1982 *Proc. Japan Acad.* A **58** 9
[13] Nimmo J J C 2006 *J. Phys. A: Math. Gen.* **39** 5053
[14] Gelfand I M and Retakh V S 1991 *Funktsional. Anal. i Prilozhen.* **25** 13
[15] Sato M 1981 *RIMS Kokyuroku* **439** 30
[16] Gelfand I, Gelfand S, Retakh V and Wilson R L 2005 *Adv. Math.* **193** 56
[17] Matveev V B 1998 The bispectral problem *CRM Proc. Lecture Notes (Montreal, PQ, 1997)* (Providence, RI: American Mathematical Society) vol 14 pp 211–26
[18] Gilson C R and Nimmo J J C 2007 On a direct approach to quasideterminant solutions of a noncommutative modified KP equation (in preparation)